Renormalization of Long-Wavelength Magnons

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The renormalization of the spin-wave frequencies in a ferromagnet, due to the thermal population of other spin waves, has been calculated taking explicit account of the effects of the dipolar coupling between the spins. The results obtained by using the Green function decoupling approximation of Tyablikov differ markedly from those obtained using the symmetric decoupling recently suggested by Callen. For purposes of comparison, the renormalization of the mode of uniform precession specifically is examined in the low-temperature limit, where it is found that the renormalization obtained using the Callen decoupling is identical to that obtained from spin-wave theory. Experiments which measure the spin-wave renormalization are discussed with regard to the theory.

I. INTRODUCTION

THE statistical mechanics of a Heisenberg ferromagnet have been analyzed by the method of double-time temperature-dependent Green functions by several authors.[1–7] Such treatments provide, as incidental results, the renormalization of spin-wave energies —that is, the shift in energy of a spin wave caused by the occupation of other spin-wave modes. In these calculations, the dipolar coupling between the spins is not included. *Note added in proof.* The author's attention has been called to the work of Meng Hsien-chen, Fiz. Tver. Tela **4**, 705 [translation: Soviet Phys.—Solid State **4**, 514 (1962)] who has extended the Tyablikov theory for spin $\frac{1}{2}$ by including the effect of the dipolar interaction. Since the Callen (Ref. 5) method of extending Green-function theory to higher spin is equally applicable when the dipolar interaction is included in the Hamiltonian, our results are valid for general spin for both Tyablikov and Callen decoupling. The Hamiltonian consists only of the Zeeman energy of the spins in the external magnetic field and the isotropic Heisenberg exchange interaction.

For long-wavelength spin waves, the magnitude of the dipolar interaction is of the order of, or exceeds, that of the exchange interaction (the contribution of the exchange interaction to the spin-wave energy, of course, vanishes as the wavelength approaches infinity). Thus, the dipolar interaction is an important factor in determining the energy and the renormalization of the energy of long-wavelength spin waves. It is of interest to find the effect of the dipolar interaction on the renormalization of these long-wavelength excitations since the experiments of LeCraw and Walker,[8] Weber and Tannenwald,[9] and Matcovich, Belson, Goldberg, and Haas[10] are concerned specifically with the measurement of the renormalization of just such excitations; in addition, the characteristics of these long-wavelength excitations are the subject of several other experiments (e.g., ferromagnetic resonance, parallel-pumping, Suhl instabilities).

In an attempt to include the dipolar interactions, Tyablikov[11] has introduced the classical demagnetization tensor into the Hamiltonian, thereby taking account of the demagnetizing effects of the surface of the finite sample. However, the local demagnetizing fields have not been included. These local demagnetizing fields play an important role since they are responsible for the removal of the spin-wave degeneracy, thus, producing the familiar spin-wave band.

We have used the full dipolar Hamiltonian to calculate the renormalization of the spin-wave energies. We find that the results we obtain by using the Tyablikov method[2] of decoupling (or the random-phase approximation) differ markedly from those we obtain using the symmetric decoupling recently suggested by Callen.[5] The results for the renormalization of the mode of uniform precession are examined in the low-temperature limit and compared with the results of spin-wave theory. The results obtained from the Callen decoupling[5] are identical to the spin-wave results, but extend these results through the entire temperature range. The LeCraw-Walker experiment[8] is discussed in terms of our results and we find that these measurements lead to a determination of a renormalization factor which is very nearly the "universal" (i.e., independent of wave vector) renormalization factor of the simple (dipolar interaction not included) spin waves.

Finally, we note that for short-wavelength spin waves

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the strength of the exchange interaction is of the order of 1000 times that of the dipolar interaction. Thus, the dipolar interaction makes a negligible contribution to the energy of short-wavelength spin waves. Since the number of shorter wavelength spin waves far exceeds the number of long-wavelength spin waves, the shorter wavelength excitations dominate the temperature dependence of the magnetization at all but very low temperatures; in addition, they determine the Curie temperature. Therefore, the previous calculations, which were concerned mainly with these thermodynamic quantities, were quite justified in ignoring the dipolar interaction. However, we emphasize that the dipolar interaction does make an important contribution to the renormalization of the long-wavelength spin waves and, hence, to experiments which probe the particular characteristics of these modes rather than sensing thermodynamic averages over all modes.

II. THE GREEN FUNCTION EQUATION

We consider the Hamiltonian

$$\mathcal{H}=-\mu H_0\sum_g S_g{}^z-\sum_{g,f}\mathbf{S}_g\cdot\mathbf{S}_f+\sum_{f,g}D_{fg}[\mathbf{S}_g\cdot\mathbf{S}_f-3(\boldsymbol{\alpha}_{fg}\cdot\mathbf{S}_f)(\boldsymbol{\alpha}_{fg}\cdot\mathbf{S}_g)],\quad(1)$$

where μS is the magnetic moment per ion; H_0 is the applied magnetic field which we assume to be in the negative z direction; S_g is the spin operator for the ion on site g; $J(g-f)$ is the exchange integral between ions at sites g and f; $D_{fg}=\mu^2/2|r_{gf}|^3$ for a classical electromagnetic dipolar interaction; r_{fg} is the distance between the sites f and g; $\boldsymbol{\alpha}_{fg}$ is the unit vector from site f to site g. In order to simplify the calculation, we consider a sample which is an ellipsoid of revolution, coaxial with the applied magnetic field (in the z direction). In addition, we assume the crystal structure to be simple, body-centered, or face-centered cubic, with one of the cubic axes oriented along the z direction.

The formalism we present follows closely that of Callen.[5] Consequently, we consider the Green function

$$G_t{}^{a+-}(g,l)\equiv\langle\langle S_g{}^+(t)\,;\,e^{aS_l{}^z}S_l{}^-\rangle\rangle,\quad(2)$$

where $S^\pm=S^x\pm iS^y$.

The Fourier transform (with respect to the time) of the Green function $G_t{}^{a+-}(g,l)$ is denoted by

$$G_E{}^{a+-}(g,l)=\langle\langle S_g{}^+;\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E,\quad(3)$$

where $E=\hbar\omega$. The equation of motion for $G_E{}^{a+-}(g,l)$ is then

$$EG_E{}^{a+-}(g,l)=\frac{1}{2\pi}\langle[S_g{}^+,e^{aS_l{}^z}S_l{}^-]\rangle+\langle\langle[S_g{}^+,\mathcal{H}];\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E,\quad(4)$$

where the square brackets denote a commutator and the single angular brackets denote an average with respect to the canonical density matrix at temperature T.

The substitution into Eq. (4) of the expression obtained for the commutator of $S_g{}^+$ with the Hamiltonian yields[12]

$$\begin{aligned}EG_E{}^{a+-}(g,l)=\frac{1}{2\pi}\Theta(a)\delta_{g,l}+\mu H_0G_E{}^{a+-}(g,l)-2\sum_f J(g-f)\langle\langle(S_g{}^zS_f{}^+-S_f{}^zS_g{}^+);\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E\\+2\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)\langle\langle S_g{}^zS_f{}^+;\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-(1-3\alpha_{fg}^{z\,2})\langle\langle S_g{}^+S_f{}^z;\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E\\-\tfrac{3}{2}\alpha_{fg}{}^{+2}\langle\langle S_g{}^zS_f{}^-;\,e^{aS_l{}^z}S_l{}^-\rangle\rangle_E],\quad(5)\end{aligned}$$

where

$$\Theta(a)=\langle[S^+,e^{aS^z}S^-]\rangle.\quad(6)$$

Since Eq. (5) for $G_E{}^{a+-}(g,l)$ involves higher order Green functions, it is necessary to choose a decoupling approximation. Tyablikov[2] has chosen a method of decoupling which ignores fluctuations in $S_g{}^z$; he therefore replaces $S_g{}^z$ by its average value, and

$$\langle\langle S_g{}^zS_f{}^+;\quad\rangle\rangle_E\underset{f\neq g}{\longrightarrow}\langle S_g{}^z\rangle\langle\langle S_f{}^+;\quad\rangle\rangle_E.\quad(7)$$

Callen[5] has recently suggested a symmetric method of decoupling which does take account of fluctuations in $S_g{}^z$. He proposes then that

$$\begin{aligned}\langle\langle S_g{}^zS_f{}^+;\quad\rangle\rangle_E\underset{f\neq g}{\longrightarrow}\langle S_g{}^z\rangle\langle\langle S_f{}^+;\quad\rangle\rangle_E\\-\alpha\langle S_g{}^-S_f{}^+\rangle\langle\langle S_g{}^+;\quad\rangle\rangle_E\\-\alpha\langle S_g{}^+S_f{}^+\rangle\langle\langle S_g{}^-;\quad\rangle\rangle_E.\quad(8)\end{aligned}$$

Note that while the third term on the right side of (8) vanishes[5] in the absence of the dipolar interaction, this is not the case when the dipolar interaction is included.

On the basis of physical requirements arising from the behavior of $\langle S^z\rangle$, Callen chooses the decoupling parameter α as

$$\alpha=\frac{1}{2S}\frac{\langle S^z\rangle}{S}.\quad(9)$$

With the inclusion of the dipolar interaction, the maximum value of $\langle S^z\rangle$ deviates[13] slightly from S. [See

[12] The additional sums

$$2\sum_f D_{fg}\{2\alpha_{fg}{}^+\alpha_{fg}{}^zS_g{}^zS_f{}^z-\alpha_{fg}{}^+\alpha_{fg}{}^zS_g{}^+S_f{}^--\alpha_{fg}{}^-\alpha_{fg}{}^zS_g{}^+S_f{}^+\}$$

which should appear on the right side of Eq. (5) have been omitted since they vanish under the symmetry assumptions we have made above.

[13] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

Sec. III, Eq. (41)]. It is, therefore, plausible in this case to choose α as

$$\alpha=\frac{1}{2\langle S^z\rangle_{\max}}\frac{\langle S^z\rangle}{\langle S^z\rangle_{\max}}. \tag{10}$$

We shall decouple the Green function equations according to the approximation given in Eq. (8) and carry the quantity α explicitly throughout the calculation. The substitution of $\alpha=0$ will then give the result as obtained by Tyablikov decoupling while the substitution of α as given by Eq. (10) will give the Callen decoupling results.

Insertion of the decoupling approximation [Eq. (8)] into Eq. (5) yields

$$\begin{aligned}
EG_E{}^{a+-}(g,l)={}&\frac{\Theta(a)}{2\pi}\delta_{g,l}+\gamma\hbar H_0G_E{}^{a+-}(g,l)-2\langle S^z\rangle\sum_f J(g-f)[\langle\langle S_f{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\langle\langle S_g{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E]\\
&+2\alpha\sum_f J(g-f)[\langle S_g{}^-S_f{}^+\rangle\langle\langle S_g{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\langle S_f{}^-S_g{}^+\rangle\langle\langle S_f{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E]\\
&+2\alpha\sum_f J(g-f)[\langle S_g{}^+S_f{}^+\rangle\langle\langle S_g{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\langle S_g{}^+S_f{}^+\rangle\langle\langle S_f{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E]\\
&+2\langle S^z\rangle\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)\langle\langle S_f{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-(1-3(\alpha_{fg}{}^z)^2)\\
&\qquad\qquad\times\langle\langle S_g{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\tfrac{3}{2}\alpha_{fg}{}^{+2}\langle\langle S_f{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E]\\
&-2\alpha\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)\langle S_g{}^-S_f{}^+\rangle\langle\langle S_g{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-(1-3(\alpha_{fg}{}^z)^2)\\
&\qquad\qquad\times\langle S_g{}^+S_f{}^-\rangle\langle\langle S_f{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\tfrac{3}{2}\alpha_{fg}{}^{+2}\langle S_g{}^+S_f{}^-\rangle\langle\langle S_g{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E]\\
&-2\alpha\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)\langle S_g{}^+S_f{}^+\rangle\langle\langle S_g{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-(1-3(\alpha_{fg}{}^z)^2)\\
&\qquad\qquad\times\langle S_g{}^+S_f{}^+\rangle\langle\langle S_f{}^-;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E-\tfrac{3}{2}\alpha_{fg}{}^{+2}\langle S_g{}^-S_f{}^-\rangle\langle\langle S_g{}^+;e^{aS_l{}^z}S_l{}^-\rangle\rangle_E].
\end{aligned}\tag{11}$$

Since there is translational invariance we consider the Fourier transforms

$$G_E{}^{a+-}(\mathbf{k})=\sum_{\mathbf{g}-\mathbf{l}}e^{-i\mathbf{k}\cdot(\mathbf{g}-\mathbf{l})}G_E{}^{a+-}(g,l), \tag{12}$$

$$J(\mathbf{k})=\sum_{\mathbf{g}-\mathbf{l}}e^{-i(\mathbf{g}-\mathbf{l})\cdot\mathbf{k}}J(g-l), \tag{13}$$

$$\psi^{-+}(\mathbf{k},a)=\sum_{\mathbf{g}-\mathbf{l}}e^{-i(\mathbf{g}-\mathbf{l})\cdot\mathbf{k}}\langle e^{aS_l{}^z}S_l{}^-S_g{}^+\rangle, \tag{14}$$

where $(\mathbf{g}-\mathbf{l})\cdot\mathbf{k}$ denotes the vector product $\mathbf{r}_{gl}\cdot\mathbf{k}$.

From (11), (12), (13), and (14), we find

$$\begin{aligned}
&\Big\{E-\gamma\hbar H_0-2\langle S^z\rangle[J(0)-J(\mathbf{k})]-\frac{2\alpha}{N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{-+}(\mathbf{k}',0)\\
&-2\langle S^z\rangle[\sum_f D_{fg}(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}}-\sum_f D_{fg}(1-3(\alpha_{fg}{}^z)^2)+\frac{2\alpha}{N}\sum_{\mathbf{k}'}\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{-+}(\mathbf{k}',0)\\
&-(1-3(\alpha_{fg}{}^z)^2)e^{i(\mathbf{g}-\mathbf{f})\cdot(\mathbf{k}'-\mathbf{k})}\psi^{-+}(\mathbf{k}',0)-\tfrac{3}{2}\alpha_{fg}{}^{+2}e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{--}(\mathbf{k}',0)]\Big\}G_E{}^{a+-}(\mathbf{k})\\
&-\Big\{\frac{2\alpha}{N}\sum_{k'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{++}(\mathbf{k}',0)-3\langle S^z\rangle\sum_f D_{fg}\alpha_{fg}{}^{+2}e^{i(\mathbf{f}-\mathbf{g})\cdot\mathbf{k}}\\
&-\frac{2\alpha}{N}\sum_{\mathbf{k}'}\sum_f D_{fg}[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{++}(\mathbf{k}',0)-(1-3(\alpha_{fg}{}^z)^2)e^{i(\mathbf{g}-\mathbf{f})\cdot(\mathbf{k}'-\mathbf{k})}\psi^{++}(\mathbf{k}',0)\\
&\qquad\qquad-\tfrac{3}{2}\alpha_{fg}{}^{+2}e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{-+}(\mathbf{k}',0)]\Big\}G_E{}^{a--}(\mathbf{k})=\frac{\Theta(a)}{2\pi},
\end{aligned}\tag{15}$$

where

$$\psi^{\pm\pm}(\mathbf{k},a)=\sum_{\mathbf{g}-\mathbf{l}} e^{-i(\mathbf{g}-\mathbf{l})\cdot\mathbf{k}}\langle e^{aS_l^z}S_l^{\pm}S_g^{\pm}\rangle\,, \tag{16}$$

$$G_E^{a--}(\mathbf{k})=\sum_{\mathbf{g}-\mathbf{l}} e^{-i(\mathbf{g}-\mathbf{l})\cdot\mathbf{k}}G_E^{a--}(g,l)\,, \tag{17}$$

$$G_E^{a--}(g,l)=\langle\langle S_g^-(t)\,;\,e^{aS_l^z}S_l^-\rangle\rangle_E\,. \tag{18}$$

By beginning with the equation of motion for the Green function $G_E^{a--}(g,l)$ and proceeding in the manner outlined above we obtain the following equation which, like Eq. (15), relates $G_E^{a--}(\mathbf{k})$ and $G_E^{a+-}(\mathbf{k})$.

$$\begin{aligned}
&\Big\{E+\gamma\hbar H_0+2\langle S^z\rangle[J(0)-J(\mathbf{k})]+\frac{2\alpha}{N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{-+}(\mathbf{k}',0)\\
&+2\langle S^z\rangle\Big[\sum_f D_{fg}(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{f}-\mathbf{g})\cdot\mathbf{k}}-\sum_f D_{fg}(1-3(\alpha_{fg}{}^z)^2)\Big]-\frac{2\alpha}{N}\sum_{\mathbf{k}'}\sum_f D_{fg}\big[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{-+}(\mathbf{k}',0)\\
&-(1-3(\alpha_{fg}{}^z)^2)e^{i(\mathbf{g}-\mathbf{f})\cdot(\mathbf{k}'-\mathbf{k})}\psi^{-+}(\mathbf{k}',0)-\tfrac{3}{2}\alpha_{fg}{}^{-2}e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{++}(\mathbf{k}',0)\big]\Big\}G_E^{--a}(\mathbf{k})\\
&+\Big\{\frac{2\alpha}{N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{--}(\mathbf{k}',0)-3\langle S^z\rangle\sum_f D_{fg}\alpha_{fg}{}^{-2}e^{i(\mathbf{f}-\mathbf{g})\cdot\mathbf{k}}\\
&-\frac{2\alpha}{N}\sum_{\mathbf{k}'}\sum_f D_{fg}\big[(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{--}(\mathbf{k}',0)-(1-3(\alpha_{fg}{}^z)^2)e^{i(\mathbf{g}-\mathbf{f})\cdot(\mathbf{k}'-\mathbf{k})}\psi^{--}(\mathbf{k}',0)\\
&\qquad-\tfrac{3}{2}\alpha_{fg}{}^{-2}e^{i(\mathbf{g}-\mathbf{f})\cdot\mathbf{k}'}\psi^{-+}(\mathbf{k}',0)\big]\Big\}G_E^{a+-}(\mathbf{k})=0\,.
\end{aligned} \tag{19}$$

In order to obtain explicit results we consider two particular cases: first, that of $k\neq 0$ spin waves and secondly, that of the mode of uniform precession ($k=0$ mode).

III. RESULTS FOR $k\neq 0$ SPIN WAVES

The dipolar sums which appear in (15) and (19) are readily evaluated if it is assumed that the sums are independent of the position $\mathbf{r}_g$ of the gth ion. This is a valid assumption for ions such that the distance to the sample surface is large compared to the excitation wavelength. For wavelengths small compared to the sample dimensions, this condition is satisfied for the large majority of the ions. Then for the case of the classical electromagnetic dipolar interaction

$$\sum_f D_{fg}(1-3(\alpha_{fg}{}^z)^2)e^{i\mathbf{k}\cdot(\mathbf{f}-\mathbf{g})}=(\gamma\hbar)^2N(4\pi/3)(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\,, \tag{20}$$

$$\sum_f D_{fg}(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)e^{i\mathbf{k}\cdot(\mathbf{f}-\mathbf{g})}=-(\gamma\hbar)^2N(2\pi/3)(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\,, \tag{21}$$

$$3\sum_f D_{fg}(\alpha_{fg}{}^{\pm})^2e^{i\mathbf{k}\cdot(\mathbf{f}-\mathbf{g})}=-(\gamma\hbar)^2N2\pi\sin^2\theta_{\mathbf{k}}\,e^{\pm 2i\phi_{\mathbf{k}}}\,, \tag{22}$$

where N is the number of spins per unit volume, $\theta_{\mathbf{k}}$ is the polar angle of the $\mathbf{k}$th spin wave with respect to the z direction, and $\phi_{\mathbf{k}}$ is the azimuthal angle of the $\mathbf{k}$th spin wave.

The dipolar sums for $k=0$ can be directly related to the demagnetization factors, so that

$$\sum_f D_{fg}(1-3\alpha_{fg}{}^{z2})=-(\gamma\hbar)^2 2\pi N(\tfrac{1}{3}-N_z)\,, \tag{23}$$

$$\sum_f D_{fg}(1-\tfrac{3}{2}\alpha_{fg}{}^+\alpha_{fg}{}^-)=-(\gamma\hbar)^2 2\pi N\left(\frac{1}{3}-\frac{N_x+N_y}{2}\right), \tag{24}$$

$$3\sum_f D_{fg}(\alpha_{fg}{}^{\pm})^2=-(\gamma\hbar)^2 2\pi N(N_x-N_y)\,. \tag{25}$$

Since we are considering the case of an ellipsoid of revolution, $N_x=N_y\equiv N_t$ and the sum in (25) is zero.

The above expressions for the dipolar sums can be substituted into Eqs. (15) and (19) and the solutions obtained. However, it is convenient before formally obtaining the solutions to recognize the $\phi_{\mathbf{k}}$ dependence of $\psi^{-+}(\mathbf{k},0)$, $\psi^{++}(\mathbf{k},0)$, and $\psi^{--}(\mathbf{k},0)$. As will be corroborated later [see Eqs. (31) and (32)], $\psi^{-+}(\mathbf{k},0)$ is independent of $\phi_{\mathbf{k}}$ while $\psi^{++}(\mathbf{k},0)$ and $\psi^{--}(\mathbf{k},0)$ vary as $e^{+2i\phi_{\mathbf{k}}}$ and $e^{-2i\phi_{\mathbf{k}}}$, respectively. This $\phi_{\mathbf{k}}$ dependence is to be expected since $\psi^{-+}(\mathbf{k},0)$ measures the average of the correlation of the transverse magnetization while $\psi^{++}(\mathbf{k},0)$ and $\psi^{--}(\mathbf{k},0)$ measure the ellipticity of the correlation of the transverse magnetization. Some of the summations over $\mathbf{k}'$ which occur in (15) and (19) contain factors of $e^{\pm 2i\phi_{\mathbf{k}'}}$. These summations vanish because of the assumed symmetry. Thus, (15) and (19) can be rewritten in the form

$$(E-A_{\mathbf{k}})G_E^{a+-}(\mathbf{k})-B_{\mathbf{k}}e^{2i\phi_{\mathbf{k}}}G_E^{a--}(\mathbf{k})=\frac{\Theta(a)}{2\pi}\,, \tag{23}$$

$$(E+A_{\mathbf{k}})G_E^{a--}(\mathbf{k})+B_{\mathbf{k}}e^{-2i\phi_{\mathbf{k}}}G_E^{a+-}(\mathbf{k})=0\,, \tag{24}$$

where

$$A_{\mathbf{k}}=\gamma\hbar H_0+2\langle S^z\rangle[J(0)-J(\mathbf{k})]+\frac{2\alpha}{N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{-+}(\mathbf{k}',0)+(\gamma\hbar)^2N\langle S^z\rangle(2\pi\sin^2\theta_{\mathbf{k}}-4\pi N_z)$$
$$-2\alpha(\gamma\hbar)^2\sum_{\mathbf{k}'}(\pi\sin^2\theta_{\mathbf{k}'}+2\pi\sin^2\theta_{\mathbf{k}'-\mathbf{k}}-2\pi)\psi^{-+}(\mathbf{k}',0)-\alpha(\gamma\hbar)^2\sum_{\mathbf{k}'}2\pi\sin^2\theta_{\mathbf{k}'}e^{2i\phi_{\mathbf{k}'}}\psi^{--}(\mathbf{k}',0)\,,\quad(25)$$

$$B_{\mathbf{k}}=(\gamma\hbar)^2N\langle S^z\rangle2\pi\sin^2\theta_{\mathbf{k}}\,.\quad(26)$$

Solving (23) and (24) for $G_E^{a+-}(\mathbf{k})$ and $G_E^{a--}(\mathbf{k})$ yields

$$G_E^{a+-}(\mathbf{k})=\frac{\Theta(a)}{4\pi E_{\mathbf{k}}}\left\{\frac{E_{\mathbf{k}}+A_{\mathbf{k}}}{E-E_{\mathbf{k}}}+\frac{E_{\mathbf{k}}-A_{\mathbf{k}}}{E+E_{\mathbf{k}}}\right\},\quad(27)$$

$$G_E^{a--}(\mathbf{k})=-\frac{\Theta(a)}{4\pi E_{\mathbf{k}}}B_{\mathbf{k}}e^{-2i\phi_{\mathbf{k}}}\left\{\frac{1}{E-E_{\mathbf{k}}}-\frac{1}{E+E_{\mathbf{k}}}\right\},\quad(28)$$

where

$$E_{\mathbf{k}}=(A_{\mathbf{k}}^2-B_{\mathbf{k}}^2)^{1/2}.\quad(29)$$

The quantities $\psi^{-+}(\mathbf{k},a)$ and $\psi^{--}(\mathbf{k},a)$ [Note that $\psi^{++}(\mathbf{k},a)=\psi^{--}(\mathbf{k},a)^*$] can be obtained from the expression

$$\psi(\mathbf{k},a)=\lim_{\epsilon\to0}i\int_{-\infty}^{\infty}\frac{G_{E+i\epsilon}^a(\mathbf{k})-G_{E-i\epsilon}^a(\mathbf{k})}{e^{E/kT}-1}d\omega.\quad(30)$$

Thus,

$$\psi^{-+}(\mathbf{k},a)=\Theta(a)\left[\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}\frac{1}{e^{E_{\mathbf{k}}/kT}-1}+\frac{1}{2}\left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right)\right],\quad(31)$$

and

$$\psi^{--}(\mathbf{k},a)=-\frac{\Theta(a)B_{\mathbf{k}}e^{-2i\phi_{\mathbf{k}}}}{E_{\mathbf{k}}}\left[\frac{1}{e^{E_{\mathbf{k}}/kT}-1}+\frac{1}{2}\right].\quad(32)$$

Finally, Callen[5] has shown that

$$\langle S^z\rangle=\frac{(S-\Phi)(1+\Phi)^{2S+1}+(S+1+\Phi)\Phi^{2S+1}}{(1+\Phi)^{2S+1}-\Phi^{2S+1}},\quad(33)$$

where in our case

$$\Phi=\frac{1}{N}\sum_{\mathbf{k}}\left\{\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}\left(\frac{1}{e^{E_{\mathbf{k}}/kT}-1}\right)+\frac{1}{2}\left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right)\right\}.\quad(34)$$

From (6), we see that

$$\Theta(0)=2\langle S^z\rangle.\quad(35)$$

Thus, Eq. (29) for the energy, Eqs. (31) and (32) for the correlation functions, and Eq. (33) relating $\langle S^z\rangle$ and Φ must be solved self-consistently for $\langle S^z\rangle$ in order to obtain the temperature dependence of the various pertinent quantities (e.g., $E_{\mathbf{k}}$ and $\langle S^z\rangle$) for each of the two decoupling schemes.

At very low temperatures, Φ is very small. The temperature-dependent term is small because of the character of the Bose factor at low temperatures. The temperature-independent term is always small since the factor (A_k/E_k-1) rapidly approaches zero with increasing k; only excitations with energies less than that corresponding to a temperature of a few degrees Kelvin make any appreciable contributions to the sum and, for materials with Curie temperatures of the order of hundreds of degrees Kelvin, such excitations are a very small fraction of the total number. Then from (33), at very low temperatures,

$$\langle S^z\rangle=S-\Phi\,,\quad(36)$$

where to lowest order in temperature the quantities $A_{\mathbf{k}}$ and $E_{\mathbf{k}}$ appearing in Φ are those for 0°K.

Therefore,

$$A_{\mathbf{k}}=\gamma\hbar H_0+2\langle S^z\rangle_{\max}[J(0)-J(\mathbf{k})]+\gamma\hbar M_0(2\pi\sin^2\theta_{\mathbf{k}}-4\pi N_z)\,,\quad(37)$$
$$B_{\mathbf{k}}=\gamma\hbar M_0 2\pi\sin^2\theta_{\mathbf{k}}\,,$$

and

$$E_{\mathbf{k}}=\gamma\hbar\{H_0-4\pi M_0N_z+2\langle S^z\rangle_{\max}[J(0)-J(\mathbf{k})]\}^{1/2}\times\{H_0-4\pi M_0N_z+2\langle S^z\rangle_{\max}[J(0)-J(\mathbf{k})]+4\pi M_0\sin^2\theta_{\mathbf{k}}\}^{1/2},\quad(39)$$

where

$$M_0=\gamma\hbar N\langle S^z\rangle_{\max}.\quad(40)$$

Thus,

$$\langle S^z\rangle=S-\frac{1}{2N}\sum_{\mathbf{k}}\left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right)-\frac{1}{N}\sum_{\mathbf{k}}\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}\left(\frac{1}{e^{E_{\mathbf{k}}/kT}-1}\right).\quad(41)$$

Equation (41) with the values of $A_{\mathbf{k}}$ and $E_{\mathbf{k}}$ given by (37) and (39) is identical to the result of Holstein and Primakoff.[13] As discussed above, the quantity $(1/2N)\times\sum_{\mathbf{k}}(A_{\mathbf{k}}/E_{\mathbf{k}}-1)$ is quite small so that the deviation of $\langle S^z\rangle_{\max}$ from S is usually negligible.[13]

IV. RESULTS FOR THE MODE OF UNIFORM PRECESSION

The dipolar sums (23)–(25) appropriate for the uniform precession mode can be substituted into the Green function equations (15) and (19) and the Green functions obtained. The solutions take a particularly simple form for an ellipsoid of revolution since $G_E^{a--}(0)=0$ and

$$G_E^{a+-}(0)=\frac{\Theta(a)}{2\pi}\frac{1}{E-E_0},\quad(42)$$

where

$$E_0=\gamma\hbar[H_0+\gamma\hbar N\langle S^z\rangle4\pi(N_t-N_z)+\alpha\gamma\hbar4\pi\sum_{\mathbf{k}}(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\psi^{-+}(\mathbf{k},0)-\alpha\gamma\hbar2\pi\sum_{\mathbf{k}}\sin^2\theta_{\mathbf{k}}\,e^{2i\phi_{\mathbf{k}}}\psi^{--}(\mathbf{k},0)].\quad(43)$$

Equation (43) is, therefore, an expression for the renormalized temperature-dependent energy (or frequency) of the $k=0$ mode where $\langle S^z\rangle$, $\psi^{-+}(\mathbf{k},0)$, and $\psi^{--}(\mathbf{k},0)$ are obtained from the self-consistent solution discussed in the previous section.

We consider the low-temperature behavior of E_0 in order to compare the results of the two decoupling methods with those obtained from a spin-wave calculation. As can be seen from Eq. (41), at very low temperatures,

$$\gamma\hbar N\langle S^z\rangle = M_0 - \gamma\hbar \sum_{\mathbf{k}} \frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}\eta_{\mathbf{k}}, \qquad (44)$$

where

$$\eta_{\mathbf{k}} = \frac{1}{e^{E_{\mathbf{k}}/kT}-1}. \qquad (45)$$

The substitution of (31), (32), and (44) into (43) yields

$$E_0 = \gamma\hbar[H_0 + 4\pi M_0(N_t - N_z)] + \alpha\Theta(0)(\gamma\hbar)^2 2\pi \sum_{\mathbf{k}} \left[(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right) + \sin^2\theta_{\mathbf{k}}\frac{B_{\mathbf{k}}}{2E_{\mathbf{k}}}\right]$$
$$+ (\gamma\hbar)^2 4\pi \sum_{\mathbf{k}} \left[(N_z - N_t)\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} + \alpha\Theta(0)(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} + \alpha\Theta(0)\sin^2\theta_{\mathbf{k}}\frac{B_{\mathbf{k}}}{2E_{\mathbf{k}}}\right]\eta_{\mathbf{k}}, \qquad (46)$$

where $A_{\mathbf{k}}$, $B_{\mathbf{k}}$, and $E_{\mathbf{k}}$ have their 0°K values as given in (37), (38), and (39).

We now consider the cases of Tyablikov and Callen decoupling. In both cases we will consider the shift ΔE of E_0 from $\gamma\hbar[H_0+4\pi M_0(N_t-N_z)]$. For Tyablikov decoupling, $\alpha=0$ and

$$\Delta E_{\text{Tyablikov}} = (\gamma\hbar)^2 4\pi (N_z - N_t)\sum_{\mathbf{k}} \frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}\eta_{\mathbf{k}}. \qquad (47)$$

For Callen decoupling, at very low temperatures, $\alpha\Theta(0)=1$ and

$$\Delta E_{\text{Callen}} = (\gamma\hbar)^2 2\pi \sum_{\mathbf{k}} \left[(1-\tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}-1\right) + \sin^2\theta_{\mathbf{k}}\frac{B_{\mathbf{k}}}{2E_{\mathbf{k}}}\right]$$
$$+ (\gamma\hbar)^2 4\pi \sum_{\mathbf{k}} \left[(N_z - N_t + 1 - \tfrac{3}{2}\sin^2\theta_{\mathbf{k}})\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} + \sin^2\theta_{\mathbf{k}}\frac{B_{\mathbf{k}}}{2E_{\mathbf{k}}}\right]\eta_{\mathbf{k}}. \qquad (48)$$

The first term on the right side of (48) results in a shift in the energy from the familiar $\gamma\hbar[H_0+4\pi M_0(N_z-N_t)]$ even at 0°K. The shift apparently arises from the local demagnetizing fields induced by the zero-point oscillation. However, as discussed in the previous section, the term is very small and is usually negligible; only for a material with a Curie point of a few degrees Kelvin would this term be measurable. The second term on the right-hand side of (48) contains the Tyablikov result plus additional terms.

The spin-wave derivation of the shift of the mode of uniform precession due to the presence of other spin waves has received considerable attention.[14–17] The spin-wave result is obtained by retaining terms in the Hamiltonian fourth order in the spin-wave variables. The second-order terms in the Hamiltonian are diagonalized by the usual Holstein-Primakoff transformation;[13] these transformed coordinates are then substituted into the fourth-order terms and the expectation value of these terms calculated in order to find the shift in the $k=0$ energy. The results are found to be identical to those of Eq. (48) including the very small temperature-independent shift. This agreement further corroborates[5] the validity of the Callen decoupling in treating magnetic systems.

V. THE LECRAW-WALKER EXPERIMENT

We now consider the results of the LeCraw and Walker[8] parallel-pumping experiment in terms of our results, the conclusions being equally applicable to the Weber and Tannenwald thin-film spin-resonance experiment.[9]

For reasons which will become evident we consider the diagram of Fig. 1. Here we have shown the $\theta_k=\frac{1}{2}\pi$ magnon branch at two temperatures T_1 and T_2. We focus our attention on a particular $\frac{1}{2}\pi$ magnon with vector k. The difference in the resonance frequencies at the two temperatures, $\omega_k{}^1-\omega_k{}^2$, is a direct measure of the renormalization of this magnon. If we are interested in the change in the curvature of the spectrum rather than in the actual renormalization of a single mode, we measure the variation in the difference $\omega_k-\omega_{k\to 0}$ as a function of temperature. The monitoring of the fre-

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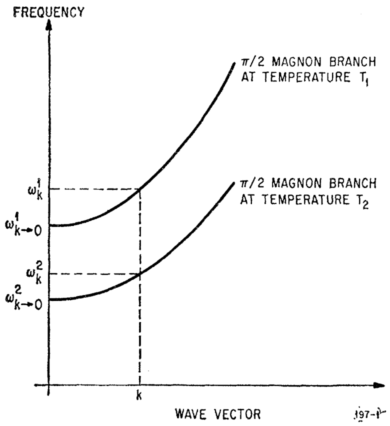

FIG. 1. Illustrative plot showing the $\frac{1}{2}\pi$ magnon branch at two temperatures T_1 and T_2 and the renormalization of a particular k magnon.

quency of a particular k magnon as a function of temperature has not as yet been achieved in practice. Thus, the renormalization of a particular k magnon has not been measured; however, the experiment of LeCraw and Walker does provide a method for measuring the change in the curvature of the magnon spectrum, or the "curvature renormalization."

LeCraw and Walker have observed the onset of instability in a parallel pumping experiment.[14,18,19] Before discussing their results with respect to renormalization theory, we briefly recall that the parallel-pump method consists of applying a microwave field parallel to the dc magnetic field. A coupling occurs between this longitudinal microwave field and the spin waves because the spin waves precess on elliptical rather than on circular cones, and hence, create components of the longitudinal magnetization which vary with twice the spin-wave frequency. In particular, the spin waves which propagate perpendicular to the dc field ($\theta_k=\frac{1}{2}\pi$) are most elliptical in their precession (because of the local demagnetization fields); hence $\frac{1}{2}\pi$ spin waves of half the pump frequency are most strongly coupled to the microwave field. The threshold for instability occurs when the rate at which energy is fed into these spin waves equals the rate at which energy is lost by these spin waves. When the microwave field exceeds the critical value, pairs of spin waves of equal and opposite wave vectors, and with frequencies equal to one-half the pump frequency, are excited. Since the critical field is dependent upon the rate at which energy is lost by the spin waves, the critical field shows a sharp maximum at that frequency which corresponds to the crossing of the $\frac{1}{2}\pi$ magnon branch and the phonon spectrum.[14,20] This magnon-phonon crossing frequency shifts with renormalization of the magnon spectrum and consequently the parallel pumping instability experiment affords a measurement of the curvature renormalization.

More specifically, consider the diagram of Fig. 2. Here we have shown the $\frac{1}{2}\pi$ magnon branch at temperatures T_1 and T_2 as well as a phonon branch which we assume to be temperature-independent (if the phonon velocity is temperature dependent this effect must be included). The sharp peak in the critical field occurs at a frequency ω_1 for temperature T_1 and at a frequency ω_2 for temperature T_2. The frequency difference $\frac{1}{2}\omega_1-\frac{1}{2}\omega_2$ is not simply the magnon renormalization since the two frequencies correspond to magnons of different wave vector. However, since the phonon frequency, velocity, and dispersion relation are known, the wave vectors k_1 and k_2 can be calculated, and, in principle, the frequency difference can be related to the renormalization of the two magnons as given through Eq. (29); the relationship is not simple.

Actually, LeCraw and Walker have done the experiment by holding the pump frequency fixed and varying the dc field. This method provides a direct measure of the temperature variation of the curvature of the $\frac{1}{2}\pi$ branch. The different dc fields result in $\frac{1}{2}\pi$ spin waves of different k being excited. As discussed above, there is a sharp increase in the critical field when the dc field is adjusted so that the unstable spin waves are degenerate with phonons of the same ω and k. The peak in the critical field and the corresponding dc field are observed as a function of temperature. Since the phonon dispersion relation is $\omega=vk$, the unstable $\frac{1}{2}\pi$ spin waves which are degenerate with the phonons always have a frequency of half the pump frequency ω_p and a corresponding $k=\omega_p/2v$. As the temperature is changed, the dc field is adjusted so that this same spin wave of wave vector k has a resonance frequency of $\frac{1}{2}\omega_p$; therefore, it is the dc field required to maintain a constant resonance frequency of a spin wave of known k which is being observed. In addition, the critical field as a function of dc field exhibits a discontinuity[14,18] at $k=0$; thus $H_{k\to 0}$, the dc field for resonance of a $\frac{1}{2}\pi$ spin wave with wave vector $k\to 0$ and with resonance frequency $\omega_p/2$, is

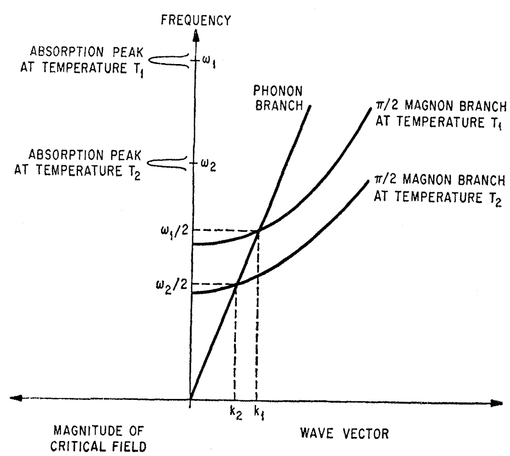

FIG. 2. Plot showing the crossing of the phonon spectrum with the $\frac{1}{2}\pi$ magnon branch at temperatures T_1 and T_2. Also indicated are the frequencies at which the peaks in the critical field occur.

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easily determined. LeCraw and Walker have measured the difference in the fields for resonance $H_{k\to 0}-H_k$ as a function of temperature.

From Eqs. (25) and (26), we see that the values of A_k and B_k for two $\theta_k=\frac{1}{2}\pi$ modes with wave vectors k and $k\to 0$ are related as

$$A_k=A_{k\to 0}-\gamma\hbar H_{k\to 0}+\gamma\hbar H_k+2\langle S^z\rangle[J(0)-J(\mathbf{k})]+\frac{\langle S^z\rangle}{NS^2}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{-+}(\mathbf{k}',0) \quad (49)$$

and

$$B_k=B_{k\to 0}, \quad (50)$$

where we have introduced the Callen value for α and neglected the very slight difference between $\langle S^z\rangle_{\max}$ and S.

Since the experiment was done in such a way that the frequencies of the two modes are identical (i.e., half the pump frequency) we find from (29), (49), and (50) that

$$H_{k\to 0}-H_k=\frac{2\langle S^z\rangle}{\gamma\hbar}[J(0)-J(\mathbf{k})]+\frac{\langle S^z\rangle}{\gamma\hbar NS^2}\times\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\psi^{-+}(\mathbf{k}',0). \quad (51)$$

Substituting Eq. (31) for $\psi^{-+}(\mathbf{k}',0)$ yields

$$H_{k\to 0}-H_k=\frac{2\langle S^z\rangle}{\gamma\hbar}\Big\{[J(0)-J(\mathbf{k})]+\frac{\langle S^z\rangle}{S^2N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\eta_{\mathbf{k}'}+\frac{\langle S^z\rangle}{S^2N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\times\left(\frac{A_{\mathbf{k}'}}{E_{\mathbf{k}'}}-1\right)(\eta_{\mathbf{k}'}+\tfrac{1}{2})\Big\}. \quad (52)$$

The third term on the right side of (52) is usually negligible compared to the first two terms. Thus, the difference in the fields for resonance is

$$H_{k\to 0}-H_k=\frac{2\langle S^z\rangle}{\gamma\hbar}\Big\{[J(0)-J(\mathbf{k})]+\frac{\langle S^z\rangle}{S^2N}\sum_{\mathbf{k}'}[J(\mathbf{k}')-J(\mathbf{k}'-\mathbf{k})]\eta_{\mathbf{k}'}\Big\}. \quad (53)$$

The right side of (53) is identical in form to the renormalization of the simple spin waves, (i.e., spin waves in the absence of the dipolar interaction) found by Callen.[5] However, the E_k and $\langle S^z\rangle$ which appear in (53) are those obtained by including the effects of the dipolar interaction. Actually, there is little difference in evaluation of the expression in (53) for the two types of spin waves over most of the temperature range and the measurement of the difference in fields for resonance is very nearly a measure of the renormalization of the simple spin waves.

In addition, for simple lattices with only nearest neighbor exchange interactions, it has been shown[21,5] that (53) can be rewritten in the form

$$H_{k\to 0}-H_k=2SR[J(0)-J(\mathbf{k})], \quad (54)$$

where the renormalization factor R is given by the expression

$$R=\frac{\langle S^z\rangle}{S}\left[1+\frac{\langle S^z\rangle}{NS^2J(0)}\sum_{\mathbf{k}'}J(\mathbf{k}')\eta(\mathbf{k}')\right]. \quad (55)$$

Here

$$J(\mathbf{k})=J\sum_{\delta}e^{i\mathbf{k}\cdot\delta}, \quad (56)$$

where δ is summed over all z nearest neighbors and $J(0)=zJ$ is the $k=0$ Fourier component of the exchange interaction. Therefore, all the simple spin waves are renormalized by the same renormalization factor independent of the k of the spin wave. This is the renormalization factor which can be obtained from the measurement of the difference in fields for resonance. It is independent of the wave vector of the particular spin wave being measured and is dependent only on the temperature.

Finally, we note that the above discussion is valid for a ferromagnet with a simple lattice. LeCraw and Walker's measurements were done on ferrimagnetic yttrium iron garnet. The detailed analysis of their data therefore must await the extension of the theory to this case.

ACKNOWLEDGMENT

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